# Flutter Analysis of Rotating Cylindrical Shells Immersed in a Circular Helical Flowfield of Air

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**The aeroelastic stability of a long, thin cylindrical shell with the outer surface exposed to an inviscid, helical flow of air is investigated. The cylinder behavior is described by classical shell equations, whereas the aerodynamic forces are described by the linearized potential theory. The approach that is used herein examines the nature of stability of the system when the system is "slightly" perturbed from its initial equilibrium state. In this paper, numerical results are presented only for the special case of swirl flow around a nonrotating shell, i.e., the axial flow velocity is set to zero. These results indicate that traveling wave type of flutter can be caused by coalescence of backward and forward traveling waves. Two approximate theories are presented and the results are compared.**

## Nomenclature

$a$ = radius of cylindrical shell
$b$ = radius of outer shell
$C$ = velocity of sound in air
$C_1$ = $\rho a^2(1 - \nu^2)/E$
$C_0$ = velocity of sound at the surface of shell
$C_s$ = velocity of sound at stagnation temperature
$e_1,e_2$ = elements of determinant as defined in (B22)
$F$ = real part of $\phi$
$f$ = frequency of oscillation of shell in cycles per second
$f_R$ = real part of $f$
$f_s$ = bench frequency of shell
$f_I$ = imaginary part of $f$
$G$ = imaginary part of $\phi$
$h$ = thickness of shell
$i$ = $(-1)^{1/2}$
$L$ = linear operator
$l$ = length of shell
$M$ = reduced frequency parameter = $a\omega/nC_0$
$M_R$ = real part of $M$
$M_I$ = imaginary part of $M$
$M_\theta$ = Mach number of flow in the tangential direction
$M_X$ = Mach number of flow in the axial direction
$M_\Omega$ = $a\Omega/nC_0$
$\bar{M}$ = dimensionless torque
$m$ = number of half waves in the axial direction
$\bar{N}_x$ = dimensionless axial load on the shell
$\bar{N}_\theta$ = dimensionless radial load on the shell
$n$ = number of circumferential waves
$p$ = dynamic pressure
$\hat{p},\hat{q}$ = real and imaginary parts of the amplitude of $p$
$r$ = radial coordinate
$R$ = $r/a$
$\bar{R}$ = $b/a$
$S^2$ = $C^2/C_0^2$
$t$ = time coordinate
$u,v,w$ = displacement of an element of shell in the axial, tangential, and radial directions, respectively
$U,V,W$ = velocity of fluid in the axial, tangential, and radial directions, respectively
$U^s,V^s,W^s$ = steady-state $U$, $V$, and $W$
$V_f$ = velocity of fluid at the surface of the shell along the tangential direction
$V_f^*$ = relative velocity
$X,Y,Z$ = inertial coordinate system
$\alpha$ = $h^2/12a^2$
$\beta$ = $n\theta + \lambda\xi - \omega t$
$\xi$ = $X/a$
$\gamma$ = ratio of specific heats of air
$\omega$ = frequency of oscillation in radians/sec
$\Omega$ = angular velocity of shell in radians/sec
$\Phi$ = velocity potential
$\Phi^s$ = steady-state velocity potential
$\phi$ = perturbed velocity potential
$\theta$ = tangential coordinate
$\lambda$ = $m\pi a/l$

## Introduction

TRAVELING wave type of vibrations observed in flexible rotating shafts of thin-walled construction has been the subject of considerable interest in the recent past. Armstrong et al.,[1] Macke,[2] DiTaranto,[3] and the present author[4] have presented theories to explain the phenomenon of traveling waves in cylindrical shells rotating about their longitudinal axis. On the other hand, the problem of aeroelastic stability of thin walled cylinders has been studied by Leonard and Hedgepeth,[5] Miles,[6] Bolotin,[7] and recently by Dowell.[8] All of these references on aeroelastic stability consider only an axial flow on the surface of a nonrotating shell. The purpose of this paper is to present an analysis that determines the aeroelastic stability characteristics of a rotating cylindrical shell subjected to a compressible fluid flowfield, which has both axial and tangential velocities, i.e., the fluid flow is in the form of a circular helix. The tangential velocity is assumed to be the result of a vortex flow originating from an outer shell.

In addition to the theoretical interest in a problem of this nature, it is likely to find technical applications in the design of rotor systems of jet engines.

## Formulation

The mathematical model which forms the basis for this analytical study is shown in Fig. 1. It consists of a thin, uniform, long cylindrical shell rotating about its longitudinal axis in a compressible fluid. The fluid is assumed to be contained within an annulus and swirling around the surface of shell in the form of a vortex as it flows along the longitudinal axis. Although the outer boundary of the annulus is assumed to be rigid, the analysis can be extended to include flexibility of the outer shell. Other important assumptions are summarized: 1) The usual assumptions that are made in linear, elastic thin shell theory are applicable here. 2) The shell may be subjected to constant external pressure, constant axial force and/or torque at its ends. 3) The fluid is assumed to be a continuum and an ideal gas with constant specific heats. 4)

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The fluid is assumed to be inviscid, nonconducting and any effects such as those due to boundary layer, drag or friction are neglected. 5) The flow is assumed to be irrotational and isentropic.

The general approach that will be taken here is to examine the stability of the cylindrical panel when it is "slightly perturbed" from its equilibrium position. An initial state for the entire system serves as the reference state.

Under the assumptions previously stated, the equations that govern the motion of the fluid may be written in terms of the potential function $\Phi$ as follows[9]:

$$\left(1 - \frac{\Phi_r^2}{C^2}\right)\Phi_{rr} + \left(1 - \frac{\Phi_\theta^2}{r^2C^2}\right)\frac{\Phi_{\theta\theta}}{r^2} + \left(1 - \frac{\Phi_x^2}{C^2}\right)\Phi_{xx} - \frac{\Phi_{tt}}{C^2} + \left(1 - \frac{2r\Phi_{rt}}{C^2} + \frac{\Phi_\theta^2}{C^2r^2}\right)\frac{\Phi_r}{r} - \frac{2}{C^2r^2}(\Phi_r\Phi_{r\theta} + \Phi_{\theta t})\Phi_\theta - \frac{2}{C^2}\left(\Phi_r\Phi_{xr} + \Phi_{xt} + \frac{\Phi_\theta\Phi_{\theta x}}{r^2}\right)\Phi_x = 0 \quad (1)$$

(Subscripts denote differentiation with respect to the variables.)

A steady state (initial state for the fluid in equilibrium) for the potential $\Phi$, $\Phi = \Phi^s$ may be assumed as the vortex flow is characterized by the relation (radius $\times$ velocity) = constant. Thus,

$$\Phi^s = A\Theta + UX \quad (2)$$

From Eq. (2), it is evident that $W^s = 0$. But

$$V^s = [(1/r)\Phi_\theta{}^s] = A/r \text{ and } \qquad U^s = \Phi_x{}^s = U$$

where $U^s$, $V^s$, and $W^s$ represent the steady-state velocities in the fluid along the axial, tangential, and radial directions. The constant $A$ may be determined by prescribing a velocity for the fluid $V_f$ at the surface of the shell. Thus,

$$V = V_f{}^*; \quad r = a; \quad V_f{}^* = V_f - a\Omega$$

$$\therefore A = aV_f{}^* \quad \therefore \Phi^s = V_f{}^*a\Theta + UX$$

A "slightly perturbed" state is represented by the potential function $\Phi(r,\Theta,x,t)$, such that,

$$\Phi = \Phi^s + \phi(r,\Theta,x,t) \quad (3)$$

Substituting Eq. (3) into Eq. (1) and omitting all higher order terms, lead to the differential equation in terms of the perturbed potential function $\phi$; i.e.,

$$\phi_{rr} + \left(1 - \frac{a^2V_f{}^{*2}}{C^2r^2}\right)\frac{\phi_{\theta\theta}}{r^2} + \left(1 - \frac{U^2}{C^2}\right)\phi_{xx} - \frac{2}{C^2}\left(aV_f{}^*U\frac{\phi_{\theta x}}{r^2} + aV_f{}^*\frac{\phi_{\theta t}}{r^2} + U\phi_{xt}\right) + \left(1 + \frac{a^2}{r^2}\frac{V_f{}^{*2}}{C^2}\right)\frac{\phi_r}{r} - \frac{\phi_{tt}}{C^2} = 0 \quad (4)$$

The slightly disturbed state of the shell may be represented by the equations of motion in terms of the displacements $u,v,w$ as follows:

$$[L]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\bar{p} \end{Bmatrix} \quad (5)$$

where $\bar{p}$, the aerodynamic pressure on the shell surface, is determined by the well-known Bernoulli equation:

$$(p)_{r=a} = -\rho_f\left(\phi_t + \frac{a}{r^2}V_f{}^*\phi_\theta + U\phi_x\right)_{r=a}$$

$$\bar{p} = \left[\frac{pa^2(1-\nu^2)}{Eh}\right] \quad (6)$$

The linear operators $L_{ij}$ are shown in Appendix A.

The boundary conditions that are required to be satisfied by the system may be stated as follows:

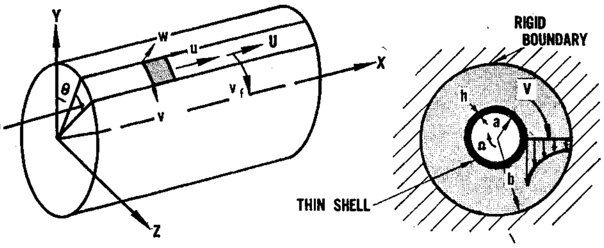


**Fig. 1 Coordinate system.**

1) At the surface of the shell, the radial velocities measured in the shell and the fluid shall be the same,

$$(\partial\phi/\partial r)_{r=a} = dw/dt \quad (7)$$

2) At the outer boundary $r = b$, the radial velocity shall be zero;

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=b} = 0 \quad (8)$$

The panel displacements $u$, $v$, and $w$ are assumed to be the following form:

$$u = \hat{u}e^{i\beta} \quad (9a)$$

$$v = \hat{v}e^{i\beta} \quad (9b)$$

$$w = \hat{w}e^{i\beta} \quad (9c)$$

$$\beta = n\Theta + \lambda\xi - \omega t \quad \lambda = m\pi a/l \quad \xi = x/a \quad (9d)$$

Consistent with the shell deflection functions is a velocity potential of the form

$$\phi = \hat{\phi}e^{i\beta}; \quad \hat{\phi} = \hat{\phi}(r) \quad (10)$$

Equations (4–10) constitute a well defined mathematical boundary-value problem which has been solved by the method presented in Appendix B.

## Numerical Solution to the Problem

The parameters that describe the shell are radius, thickness, length, material properties, axial and circumferential wave numbers, rpm of the shell, and size of the annulus. The parameters that describe the nature of the fluid are density, ratio of specific heats, and velocity of sound in the medium at a given temperature. As derived in Appendix B, for given values of these shell and fluid parameters and for prescribed axial and tangential flow velocities, the boundary-value problem recently posed is completely solved when a frequency parameter $M = a\omega/nCo$ is found such that the condition,

$$(\hat{\phi}_r)_{r=a} = D(\hat{\phi})_{r=a} \quad (11)$$

is satisfied.

It may be noted that a particular difficulty that arises at once pertains to the choice of $\omega$. $\omega$ being in general a complex number, it would seem that there is no evident guess that can be made regarding its real *and* imaginary parts. This is indeed the case if the solution is to be found beyond the onset of instability in order to determine the severity of flutter. However, prior to such an onset the imaginary part of $\omega$ may be assumed to be zero under no damping conditions. The problem would then reduce to the determination of frequencies which define a neutrally stable condition for the aeroelastic system. Such a condition may be determined as a special case of Eq. (11), i.e.,

$$(\hat{\phi}_r)_{r=a} + \frac{[n^2(1+\alpha) - \delta_1](1-\nu^2)(M_\theta - M_R)^2(\hat{\phi})_{r=a}}{\dfrac{E}{\rho_f}\dfrac{h}{a}\{[(1+\alpha) - \delta_1/n^2][1 + \alpha n^4 - \delta_1] - (1+\alpha n^2)^2\}} = 0 \quad (12)$$

where $\delta_1 = C_1 Co^2 n^2 M^2{}_R/a^2$ for the case of a nonrotating shell with no axial waves and no external loads. In such a case, the procedure is to assume a frequency and obtain a numerical solution to the differential Eq. (4) subjected to the prescribed boundary conditions. Thus $\hat{\phi}$ and $\hat{\phi}_r$ are obtained throughout the entire field $1 \leq R \leq \bar{R}$ where the number $\bar{R}$ refers to the outer boundary. $\hat{\phi}$ and $\hat{\phi}_r$ being thus available at $R = 1$, the expression (12) is checked. If it is satisfied, then the assumed frequency is the required frequency; otherwise, an increment on the frequency is assumed and the whole process repeated until the desired value which satisfies Eq. (12) is obtained.

In the general case, i.e., when the imaginary part of the frequency is not zero, the numerical work involved is rather cumbersome. The flutter condition in this case reduces to the requirement that the elements $e_1$, $e_2$ of a certain matrix $(e)$ be zero simultaneously, as explained in detail in Appendix B. $e_1,e_2$ are functions of both the real and imaginary parts of the frequency parameter. Therefore, it becomes necessary to follow a systematic procedure to pinpoint that unique combination of real and imaginary parts of the frequency which results in $e_1 = e_2 = 0$. Such a procedure is outlined below:

1) Assume the Mach number close to the critical Mach number (or close to the Mach number for which the solution was last obtained).

2) Assume the real part of the trial frequency close to the flutter frequency (or close to the frequency last obtained).

3) Assume several imaginary parts each of which is a certain percentage of the real part.

4) Compute $e_1,e_2$ and plot them as a function of the ratio (imaginary part/real part) of the frequency.

5) If $e_1$ and $e_2$ are not zero for the same ratio assume a different real part and repeat steps (2–5) until the required unique ratio is obtained.

It is necessary to observe that this procedure may be automated such that a graphical procedure may altogether be eliminated. However, a question may be posed as to the need for determinating the severity of flutter in a given design problem. All that may be required in a design analysis may be the critical Mach number and the associated frequency at which a shaft is likely to flutter under given conditions.

## Approximate Theories

### A. Slender Body Theory

A major source of difficulty in flutter analysis is the problem of predicting air forces resulting from vibrations of a structure. Several approximate theories are proposed in literature which express the dynamic pressure in terms of local displacements and their derivatives. For the problem under study it was found that the so-called Slender Body Theory gave qualitatively useful results. One of the principle advantages of using such a simple theory is that the flutter problem reduces to a characteristic value problem. Thus, an examination of the roots of the flutter determinant is enough to predict the onset of instability. A detailed discussion of the application of the Slender Body Theory to the two dimensional case ($\lambda = 0$, $U = 0$) is given.

Assuming that the shell under consideration is long and ignoring variation of any quantity along the axis, the flutter problem is completely formulated by the partial differential equations of motion shown below:

$$C_1\ddot{v} - (1 + \alpha)v^{\cdot\cdot} + 2C_1\Omega\dot{w} - w^{\cdot} + \alpha w^{\cdot\cdot\cdot} = 0 \quad \text{(13a)}$$

$$C_1\ddot{w} + w + \alpha w^{\cdot\cdot\cdot\cdot} - \bar{N}_\theta w^{\cdot\cdot} - 2C_1\Omega\dot{v} + v^{\cdot} - av^{\cdot\cdot\cdot} + \bar{N}_\theta v^{\cdot} - C_1\Omega^2 w + \bar{p}_a = 0 \quad \text{(13b)}$$

where $\alpha = h^2/12a^2$, $C_1 = \rho a^2(1 - \nu^2)/E$, $\bar{p}_a = p_a(1 - \nu^2)a^2/Eh$, $(\dot{\ }) = \partial/\partial t$, $(\ )^{\cdot} = \partial/\partial\Theta$, $\bar{N}_\theta$ = membrane stress due to rotation of shell, and $\Omega$ = angular velocity of shell in radians/sec. $\bar{p}_a$ is the aeroelastic dynamic pressure which influences and is influenced by vibrations of the shell. It is evident that an expression for this pressure in terms of $w$ and its derivatives reduces the problem to an eigen value problem. Slender Body Theory proposes the following expression for the aerodynamic pressure acting on a circular cylindrical shell whose surface is subjected to a fluid flow of velocity $U$ along its longitudinal axis:

$$p_a = \frac{\rho_f U^2 a}{n}\left(\frac{\partial}{\partial x} + \frac{1}{U}\frac{\partial}{\partial t}\right)^2 w(x,t)\ \cos n\Theta \quad (14)$$

where $n$ = number of circumferential waves and $p_f$ = density of fluid. To be applicable to the problem of a rotating shell under the action of a vortex flow of velocity $V_f$, the previous expression may be written in the form of

$$p_a = \frac{\rho_f V^2{}_f{}^* a}{n}\left(\frac{1}{a}\frac{\partial}{\partial\Theta} + \frac{1}{V_f{}^*}\frac{\partial}{\partial t}\right)^2 w(\Theta,t) \quad (15)$$

Equation (14) is an approximation to the exact expression for the pressure as shown by Dowell,[8] for incompressible flow. Dowell comments that this expression "should probably be a reasonable expression even for compressible flow at least up through low supersonic Mach numbers." Furthermore, for the axial flow problem, Dowell and Widnall,[10] show that the form of expression for the dynamic pressure reduces to the so-called Slender Body form under the following assumption: $(a/1) \to 0$, $n \neq 0$, and axial velocity component is large. No such direct reduction and therefore justification can be provided in the present problem because of the difficulty in obtaining analytical solutions to the governing equations.

For the problem under consideration where the fluid flow is a vortex around the circumference of the shell, the assumption is made that Eq. (14) is applicable in a slightly modified form, as shown in Eq. (15). The modification is needed to conform to the flow around the cylindrical shell and to take into account the rotation of the shell. Admittedly, the expression is not rigorously proven to be applicable to this problem and is, therefore, open to question. However, it was found that the numerical results have value of a qualitative nature.

The solution to Eqs. (13) is assumed to be of the form

$$v = \hat{v}e^{i\beta} \quad w = \hat{w}e^{i\beta} \quad \beta = n\Theta - \omega t \quad (16)$$

Substitution of Eq. (16) into Eq. (13) yields the desired characteristic equation as follows:

$$\bar{\omega}^4\left(1 + \frac{C_3}{n}\frac{a}{h}\right) - \bar{\omega}^3\left(2\bar{V}^{\cdots}{}_f{}^*\frac{C_3}{h}\right) - \bar{\omega}^2\left[1 + \alpha n^4 + n^2\bar{\Omega}^2 + 3\bar{\Omega}^2 + n^2(1 + \alpha)\left(1 + \frac{C_3}{n}\frac{a}{h}\right) - \frac{n}{ah}C_3\bar{V}_f{}^{*2}\right] - \bar{\omega}\left[4n\bar{\Omega}(1 + \alpha n^2) + 2n\bar{\Omega}^3 - \frac{2(1 + \alpha)n^2}{h}C_3\bar{V}_f{}^*\right] + \left[n^2(1 + \alpha)(\alpha n^4 + n^2\bar{\Omega}^2 + 1 - \bar{\Omega}^2) - n^2(1 + \alpha n^2)^2 - n^2\bar{\Omega}^2(1 + \alpha n^2) - \frac{(1 + \alpha)n^3 C_3\bar{V}_f{}^{*2}}{ha}\right] = 0 \quad (17)$$

where $\bar{\omega} = (C_1)^{1/2}\omega$, $\bar{\Omega} = (C_1)^{1/2}\Omega$, $\bar{V}_f{}^* = (C_1)^{1/2}V_f{}^*$, $C_3 = \rho_f/\rho$.

Numerical solution to the quartic in Eq. (17) is sought, and the roots expressed in the form $\omega_R + i\omega_I$ are examined. The nature of $\omega_I$ determines the onset of instability as $e^{i(n\Theta - \omega_R t - i\omega_I t)} = e^{i(n\Theta - \omega_R t)}e^{\omega_I t}$. Clearly for $\omega_I > 0$ oscillations increase exponentially with time.

### B. Incompressible Flow

At low velocities, the assumptions of incompressible flow are appropriate. Based on these assumptions an analytical criterion for obtaining the characteristic frequencies may be derived which is simple and convenient for numerical evaluation. As in Slender Body Theory, the numerical results based on incompressible flow assumptions have been obtained for the two-dimensional flow only. For low fluid velocities these results compare favorably with those obtained from the general theory presented in Appendix B.

It can be shown that the differential equation for the potential $\phi$ reduces to

$$\hat{\phi}_{RR} + \left(1 + \frac{M_\theta{}^{*2}}{S^2R^2}\right)\frac{\hat{\phi}_R}{R} + \frac{n^2}{S^2}\left[\left(M - \frac{M_\theta{}^*}{R^2}\right)^2 - \frac{S^2}{R^2}\right]\hat{\phi} = 0 \tag{18}$$

If the flow is assumed to be incompressible, the following simplifications are in order:

$$S^2 \rightarrow 1$$
$$[1 + (M_\theta{}^{*2}/S^2R^2)] \rightarrow 1 \tag{19}$$
$$[M - (M_\theta{}^{*2}/R^2)]^2 \ll 1/R^2$$

With the aforementioned simplifications, the differential Eq. (18) reduces to an equidimensional equation, i.e.,

$$\hat{\phi}_{RR} + (\hat{\phi}_R/R) - (n^2/R^2)\hat{\phi} = 0 \tag{20}$$

for which the solution may be written as

$$\hat{\phi} = C_{11}R^n + C_{12}R^{-n} \tag{21}$$

Imposing the boundary condition at $R = \bar{R}$, i.e. $\hat{\phi}'(\bar{R}) = 0$, it may be shown that $C_{11} = C_{12}\bar{R}^{-2n}$.

The equation which relates the amplitude of the shell displacement with that of $\phi$ (1) may be obtained as a special case of Eq. (B16) and may be shown to be

$$i\hat{w} = \frac{[n^2(1+\alpha) - \delta_2]\left[\dfrac{nC_0(1-\nu^2)}{E/\rho_f h/a}(M_\theta{}^* - M)\right]\hat{\phi}(1)}{\left[\begin{matrix}[n^2(1+\alpha) - \delta_2](1 - \delta_2 + \alpha n^4 + \bar{N}_\theta n^2) \\ -(n + \alpha n^3 + \delta_3)(n + \alpha n^3 + n\bar{N}_\theta + \delta_3)\end{matrix}\right]} \tag{22}$$

$$iei\hat{w} = [KnCo(1-\nu^2)/(E/\rho_f)(h/a)](M_\theta{}^* - M)\hat{\phi}(1)$$

where $\delta_2 = (C_1n^2C_0{}^2/a^2)(M^2 + M_\Omega{}^2)$

$$\delta_3 = (C_1n^2C_0{}^2/a^2)2MM_\Omega$$

The other boundary condition at $R = 1$ [see Eq. (B12)] reduces to

$$\hat{\phi}'(1) = nC_0(M_\theta{}^* - M)i\hat{w} \tag{23}$$

$$\therefore \hat{\phi}'(1) = \frac{n^2C_0{}^2(1-\nu^2)}{(E/\rho_f)(h/a)}K(M_\theta{}^* - M)^2\hat{\phi}(1) \tag{24}$$

which, in view of the solution in Eq. (21) reduces to a relatively simple criterion,

$$ie\ \frac{KnC_0{}^2(1-\nu^2)}{(E/\rho_f)(h/a)}(M - M_\theta{}^*)^2\left(\frac{1 + \bar{R}^{-2n}}{1 - \bar{R}^{-2n}}\right) + 1 = 0 \tag{25}$$

The numerical procedure is direct and simple. A frequency is assumed for given shell and fluid parameters, and expression (25) is checked. If it is zero then the assumed frequency is an eigenvalue. Otherwise a different frequency is assumed and the process is repeated until the criterion is satisfied.

As observed earlier, Slender Body Theory and Incompressible Flow Theory have been applied to the two-dimensional case only (*ie* $\lambda$ and $U$ are set to zero in the General Theory). However, it is clear that these approximations may be extended to the general case which includes axial traveling waves in the shell and axial flow of air on the surface of the shell. The results of such an extension will be presented in a subsequent paper.

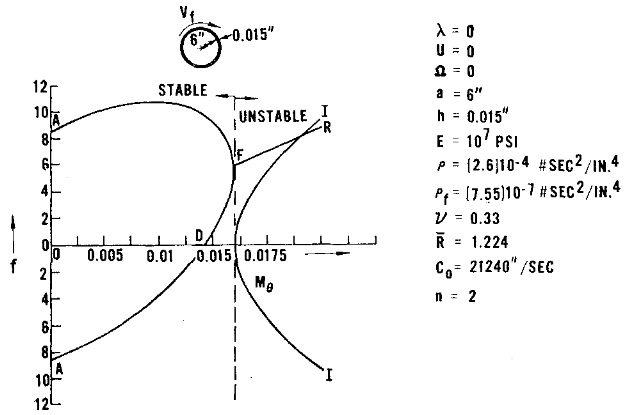


**Fig. 2 Slender body theory.**

## Results and Conclusions

As observed earlier, the use of an appropriate theory such as Slender Body Theory to express the dynamic pressure reduces the flutter problem to a standard eigenvalue problem. Thus, a characteristic equation is obtained, the solution of which defines the frequencies at which the shell will tend to be in stable, neutral, or unstable equilibrium. A plot of such frequencies is shown in Fig. 2. At Mach 0, the backward and forward waves have the same speed ($A$, $A$) and this frequency is very close to the invacuo natural frequency of the shell. Upon increasing the Mach number, there is an unsymmetric distortion of the frequencies of backward and forward waves. At a certain Mach number, denoted by the letter $D$, the speed of the backward wave has approached zero, after which it continues as a forward wave $DF$ lagging behind the forward wave $AF$. Both these waves coalesce at $F$. Up to, and including $F$, the characteristic equation yields only real roots for the wave speeds. Beyond $F$, a real part of the wave speed is always accompanied by two equal and opposite (in sign) imaginary parts. As the solution for all the shell displacements is written as a traveling wave with the factor $e^{i(\eta\theta - \omega t)}$ it is clear that $\omega_I = 0$ indicates neutrally stable condition, $\omega_I > 0$ indicates an unstable condition. The severity of instability is determined by the magnitude of $\omega_I$ at a given air speed.

Thus, a traveling wave-type of flutter caused by coalescence of backward and forward waves is predicted. The point $D$ corresponds to divergence-type of instability. It is believed that the latter may be shown to be a condition of weak instability if structural damping is included in the formulation.

Circumferential traveling wave-type of vibrations may develop in a rotor system under air flow. The resulting displacement at any location at any time is the sum of the amplitudes of backward and forward waves. Thus, backward and forward waves may be considered as independent modes of vibration, a linear combination of which provides the total solution. It may then be theorized that a flutter point such as $F$ in Fig. 2 is simply the classical coalescence point at which two modes of vibration, which are otherwise uncoupled, merge. It should be noted that the waves in the direction of flow (forward waves) tend to be unstable, and backward traveling waves always tend to be stable. This stands to reason as the influence of an increasing air velocity can be more effective and predominant on a wave in its own direction. This is believed to be the reason for the gradual reduc-

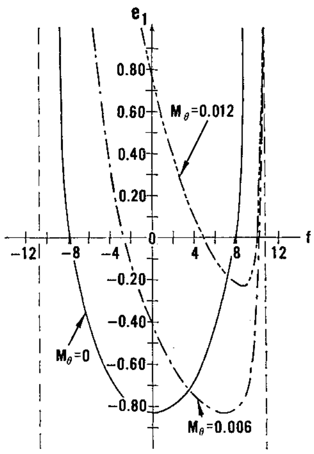


**Fig. 3** $e_1 \sim f$ **curves with** $M$ **as parameter.**

tion in the speed of backward traveling waves. As the dynamic pressure is related to air speed, an increase in the latter results in an increase in pressure on the shell surface. This phenomenon is believed to be the cause for a gradual fall in the natural frequencies. In terms of a static analogy, the flutter point $F$ may be considered to be the point at which a static instability (buckling) may be predicted.

Traveling wave-type of flutter has been previously observed and reported by Dowell[11] and Dugundji.[12] In his experimentation on thin panels, Dugundji, et al., observed that 1) "it appeared that definite flutter of a generally traveling wave-type character was obtained for this panel" and 2) "the waves generally appeared to be traveling downstream, although there seemed to be some 'sticking' or standing wave components present near the center of the panel." The agreement between theory and experiment in the prediction of the wave speed and frequency at flutter has been found to be rather poor, whereas the prediction of flutter speed was generally good. Reference 12 is recommended for more details in this regard. Computations, based on the theory presented here, indicate that the flutter frequency differs from the invacuo natural frequency of the cylinder. It was observed that this difference is smaller for larger $h/a$ ratios.

Figure 3 shows the manner in which the function $e_1$ varies as the Mach number changes. At Mach 0, $e_1$ passes through zero at a frequency very close to the invacuo natural frequency characterized by the asymptote. Upon approaching the critical Mach number, $e_1$ passes through 0 at two positive values of frequencies. Subsequently, at a Mach number very close to the critical Mach number, $e_1$ passes through zero at two frequencies which are very close to each other. Thus, the frequencies at which $e_1$ passes through zero approach each other as the Mach number approaches the flutter Mach number.

Figure 4 shows a comparison of the results obtained from the general theory with those from the approximate theories. The agreement between the general theory and incompressible flow theory is excellent. This appeared to be generally true for thin shells only (and, therefore, for low subsonic flutter conditions). Slender Body Theory, in general, predicts a higher Mach number and higher flutter frequency. This is expected in view of the fact that there is no obvious way of incorporating the effect of the cavity size in Slender Body Theory approach. Preliminary examination indicates that as the cavity size increases, the results from the general theory (and, also, the incompressible flow theory) tend towards those obtained from Slender Body Theory.

It must be noted that the results (based on Compressible Flow Theory) shown in Fig. 4 were obtained by setting the imaginary part of the frequency of oscillation to zero. With this assumption $e_2$ is identically zero and, therefore, only $e_1$ is computed. But the analysis and numerical results presented by Dowell[8] for axial flow on the surface of a cylindrical shell clearly indicate the presence of a phase shift in $\omega$ for compressible flow for all flow regimes. However, it is interesting to note from Ref. 8 that 1) the imaginary part associated with one of the two traveling waves was very small prior to coalescence and 2) the imaginary part associated with the other wave was identically zero (and, therefore, neutrally stable). For the axial flow problem the solutions can be written in an analytical form and observations previously referred to could be made. For the problem of vortex flow presented in this paper, the approach being mainly numerical, it is quite tedious to determine the imaginary parts associated with the real parts. Therefore, a further assumption is made that the imaginary parts prior to coalescence are so small that they can be neglected in computing a "practical" flutter boundary. It is intended to make some typical computations to verify this assumption. It is hoped that these computations will show the qualitative behavior of both the waves to be identical to that observed for the axial flow problem, i.e., 1) prior to coalescence, the backward traveling waves are lightly damped and the forward traveling waves are neutrally stable and 2) upon approaching coalescence the backward traveling waves tend to be strongly stable while the forward traveling waves tend to be strongly unstable.

Figure 5 illustrates the graphical procedure outlined earlier to determine the ratio $f_I/f_R$ beyond the flutter frequency. As observed before, the method is rather cumbersome. The ratio of $f_I/f_R$ (for the example problem) at which both $e_1$ and $e_2$ go to zero is found to be approximately 0.5 for an $f_R = 9.3$. Thus, it is clear that a strong instability is indicated for this shell under the given conditions.

In all these calculations $n = 2$ was chosen because the

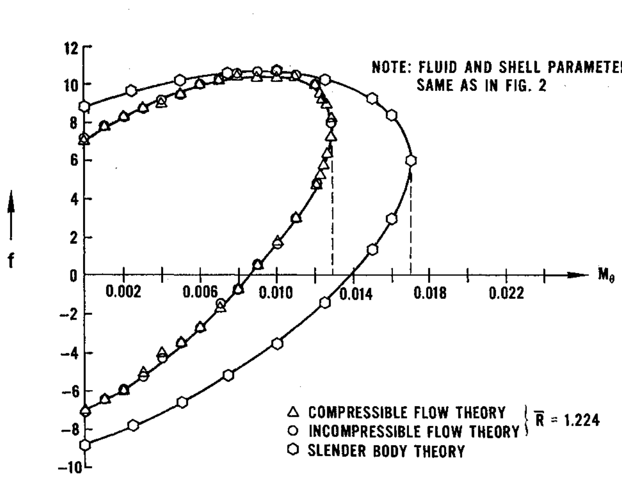


**Fig. 4 Comparison of various theories.**

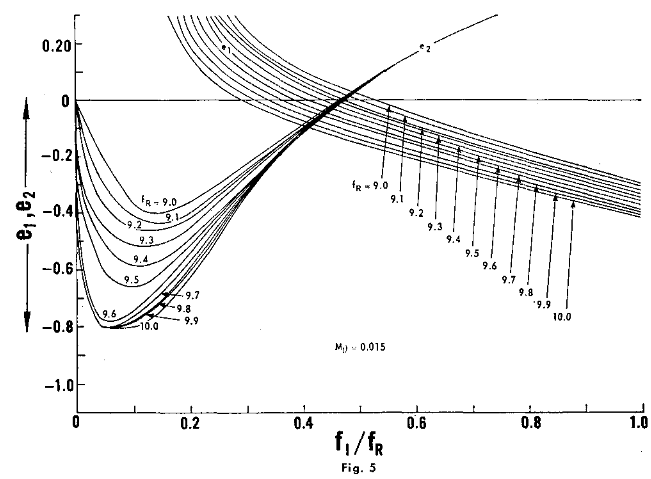


**Fig. 5 Variation of** $e_1$**,** $e_2$ **with** $f_I/f_R$**.**

natural frequency of the shell has the lowest value for this $n$. For larger values of $n$, the flutter speeds tend to be larger.

## Appendix A: Governing Equations of a Motion of a Cylindrical Shell Rotating about its Longitudinal Axis

Figure 1 shows a circular cylindrical shell and the coordinate system used in this analysis. $u$, $v$, $w$ represent the displacements of an element of shell in the axial, tangential, and radial directions. For a long, thin cylindrical shell subjected to radial dynamic pressure the dynamic behavior can be completely determined by the solution of the three differential equations of motion. These equations are written below in the form of a matrix equation and include the effects of constant axial force and torque applied at the ends of a shell.[13]

$$[L]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\bar{p} \end{Bmatrix} \quad \text{(A1)}$$

The linear operators $L_{ij}$ are represented below:

$$L_{11} = C_1 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \xi^2} - a_1 \frac{\partial^2}{\partial \Theta^2} - \bar{M} \frac{\partial^2}{\partial \xi \partial \Theta} - N_\theta \frac{\partial^2}{\partial \Theta^2}$$

$$L_{12} = -(a_2 + \bar{N}_x)\partial^2/\partial\xi\partial\Theta$$

$$L_{13} = (-\nu + \bar{N}_\theta - \bar{N}_x)\partial/\partial\xi$$

$$L_{21} = -(a_2 + \bar{N}_\theta)\partial^2/\partial\xi\partial\Theta$$

$$L_{22} = C_1\left(\frac{\partial^2}{\partial t^2} - \Omega^2\right) - (1 + \alpha)\frac{\partial^2}{\partial \Theta^2} - [a_1(1 + 2\alpha) + \bar{N}_x]\frac{\partial^2}{\partial \xi^2} - \bar{M}\frac{\partial^2}{\partial\xi\partial\Theta}$$

$$L_{23} = \left(2C1\Omega\frac{\partial}{\partial t} - \frac{\partial}{\partial \Theta}\right) - \bar{M}\frac{\partial}{\partial \xi} + \alpha\left(\frac{\partial^3}{\partial\Theta^3} + \frac{\partial^3}{\partial\xi^2\partial\Theta}\right)$$

$$L_{31} = \nu\partial/\partial\xi$$

$$L_{32} = (1 + \bar{N}_\theta)\frac{\partial}{\partial\Theta} - 2C_1\Omega\frac{\partial}{\partial t} + \bar{M}\frac{\partial}{\partial\xi} - \alpha\left(a_3\frac{\partial^3}{\partial\xi^2\partial\Theta} + \frac{\partial^3}{\partial\Theta^3}\right)$$

$$L_{33} = C_1\left(\frac{\partial^2}{\partial t^2} - \Omega^2\right) + (1 + \alpha\nabla^4) - \bar{M}\frac{\partial^2}{\partial\xi\partial\Theta} - \bar{N}_x\frac{\partial^2}{\partial\xi^2} - \bar{N}_\theta\frac{\partial^2}{\partial\Theta^2}$$

$$C_1 = \rho a^2(1 - \nu^2)/E \qquad \alpha = h^2/12a^2$$

$$a_1 = 1 - \nu/2 \qquad \bar{M} = M(1 - \nu^2)/E\pi a^2 h$$

$$a_2 = 1 + \nu/2 \qquad \bar{N}_\theta = N_\theta(1 - \nu^2)/Eh$$

$$a_3 = 2 - \nu \qquad \bar{N}_x = N_x(1 - \nu^2)/Eh$$

## Appendix B: Method of Solution

The differential equation in terms of the perturbed potential function $\phi$ was shown in Eq. (4). The velocity of sound $C$ in the fluid medium is not constant. It is usually expressed in terms of the velocity at stagnation temperature and potential function as

$$C^2 = C_s^2 - [(\gamma - 1)/2][\Phi_r^2 + (\Phi_\theta^2/r^2) + \Phi_x^2] \quad \text{(B1)}$$

where $\gamma$ is the ratio of specific heats: i.e.,

$$C^2 = C_s^2 - [(\gamma - 1)/2](U^2 + V^2 + W^2) \quad \text{(B2)}$$

Letting $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ be the perturbed velocities in the axial, tangential, and radial directions,

$$\begin{aligned} U &= U^s + \tilde{U} \\ V &= V^s + \tilde{V} \\ W &= W^s + \tilde{W} \end{aligned} \quad \text{(B3)}$$

With (B3) $C^2$ may be shown to be

$$C^2/C_0^2 = 1 + [(\gamma - 1)/2]M_\theta^{*2}[1 - (1/R^2)] \quad \text{(B4)}$$

where $C_0 = C$ at the surface of the shell.

Letting $M_\theta^* = V_f^*/C_0$; $R = r/a$, $\xi = x/a$, $S^2 = C^2/C_0^2$ and $M_x = U/C_0$, the differential equation may be shown to reduce to

$$\phi_{RR} + \left(1 - \frac{M_\theta^{*2}}{S^2R^2}\right)\frac{\phi_{\theta\theta}}{R^2} + \left(1 - \frac{M_x^2}{S^2}\right)\phi_{\xi\xi} - \frac{2}{S^2}\left(\frac{M_\theta^* M_x}{R^2}\phi_{\theta\xi} + \frac{M_\theta^* a}{R^2 C_0}\phi_{\theta t} + \frac{M_x a}{C_0}\phi_{\xi t}\right) + \frac{\phi_R}{R} + \frac{1}{R^3}\frac{M_\theta^{*2}}{S^2}\phi_R - \frac{a^2}{S^2C_0^2}\phi_{tt} = 0 \quad \text{(B5)}$$

With $\phi = \hat{\phi}(R)e^{i\beta}$, Eq. (B5) reduces to

$$\hat{\phi}_{RR} + \left(1 + \frac{M_\theta^{*2}}{S^2R^2}\right)\frac{\hat{\phi}_R}{R} + \left\{\left[(nM + \lambda M_x) - \frac{nM_\theta^*}{R^2}\right]^2\frac{1}{S^2} - \left(\lambda^2 + \frac{n^2}{R^2}\right)\right\}\hat{\phi} = 0 \quad \text{(B6)}$$

where $M$, the Mach number of the traveling wave is defined as $M = a\omega/nC_0$. $M$ being in general complex ($M = M_R + iM_I$) Eq. (B6) is an ordinary linear differential equation with variable, complex coefficients, i.e., it is of the form

$$\hat{\phi}_{RR} + A(R)\hat{\phi}_R + \{B(R) + iC(R)\}\hat{\phi} = 0 \quad \text{(B7)}$$

where

$$A(R) = [1 + (M_\theta^{*2}/S^2R^2)]1/R$$

$$B(R) = \left\{\left[(M_R^2 - M_I^2)n^2 + \lambda^2 M_x^2 + 2\lambda n M_R M_x + n^2\frac{M_\theta^{*2}}{R^2} - 2\frac{n}{R^2}(nM_R + \lambda M_x)M_\theta^*\right]\frac{1}{S^2} - \left(\lambda^2 + \frac{n^2}{R^2}\right)\right\} \quad \text{(B8)}$$

$$C(R) = \frac{2M_I}{S^2}\left(n^2 M_R + \lambda n M_x - \frac{n^2}{R^2}M_\theta^*\right)$$

The pressure-velocity potential relation, in accordance with Bernoulli's equation reads as

$$(p)_{r=a} = -\rho_f[\phi_t + V_f^*(a/r^2)\phi_\theta + U\phi_x]_{r=a} \quad \text{(B9)}$$

which may be shown to reduce to

$$\hat{p} = -i\rho_f C_0 n/a[M_\theta^* - M - (\lambda/n)M_x]\hat{\phi}(1) \quad \text{(B10)}$$

where $\hat{p}$ is the amplitude of the dynamic pressure and $\hat{\phi}(1)$ represents the amplitude of the perturbed potential at $r = a$. For use with the shell equations $\bar{p}$ is needed. The latter is obtained as

$$\hat{\bar{p}} = [\hat{p}a^2(1 - \nu^2)]/Eh$$

$$\therefore \hat{\bar{p}} = \{[-inC_0(1 - \nu^2)]/E/\rho_f\, h/a\}[M_\theta^* - M - (\lambda/n)M_x]\hat{\phi}(1) \quad \text{(B11)}$$

The boundary condition at $r = a$ is

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = \frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{V_f^*}{a}\frac{\partial w}{\partial \Theta} + U\frac{\partial w}{\partial x}$$

$$\therefore (\hat{\phi}_R)_{R=1} = inC_0\left(M_\theta^* - M - \frac{\lambda}{n}M_x\right)\hat{w} \quad \text{(B12)}$$

At

$$r = b \quad (\hat{\phi}_R)_{R=\bar{R}} = 0 \quad \text{(B13)}$$

Substituting the assumed solution $u = \hat{u}e^{i\beta}$, $v = \hat{v}e^{i\beta}$, and $w = \hat{w}e^{i\beta}$, the equations of motion for the shell (A1) may be written as

$$[A + iB]\begin{Bmatrix}\hat{u}_R + i\hat{u}_I \\ \hat{v}_R + i\hat{v}_I \\ \hat{w}_R + i\hat{w}_I\end{Bmatrix} = \begin{Bmatrix}0 \\ 0 \\ -\bar{p}\end{Bmatrix}$$

Let (B14)

$$[C + iD] = [A + iB]^{-1}$$

Then $\hat{w} = \hat{w}_R + i\hat{w}_I = (C_{33} + D_{33})(-\hat{\bar{p}})$ where the subscripts 33 refer to the element in the third row and third column of the inverse matrix. Thus,

$$i\hat{w} = -(C_{33} + D_{33})i\hat{\bar{p}} \quad \text{(B15)}$$

With (B11), $i\hat{w}$ may be written as

$$iw = \frac{-nC_0(1-\nu^2)}{(E/\rho_f)(h/a)}(C_{33} + iD_{33})\left(M_\theta^* - M - \frac{\lambda}{n}M_x\right)\hat{\phi}(1) \quad \text{(B16)}$$

Furthermore, with (B12), $\hat{\phi}_R(1)$ and $\hat{\phi}(1)$ may be related as follows:

$$\hat{\phi}_R(1) + \frac{n^2C_0^2(1-\nu^2)}{(E/\rho)({}_fh/a)}(C_{33} + iD_{33})\left(M_\theta^* - M - \frac{\lambda}{n}M_x\right)^2 \times \hat{\phi}(1) = 0 \quad \text{(B17)}$$

Equation (B17) may, therefore, be written as

$$(\hat{\phi}_R)_{R=1} = D(\hat{\phi})_{R=1} \quad \text{(B18)}$$

where $D = D_R + iD_I$, which is the required condition to be satisfied at $R = 1$.

$$\begin{aligned} D_R &= \epsilon[C_{33}(M_R^{*2} - M_I^2) + 2D_{33}M_R^*M_I] \\ D_I &= \epsilon[D_{33}(M_R^{*2} - M_I^2) - 2C_{33}M_R^*M_I] \end{aligned} \quad \text{(B19)}$$

$$\epsilon = \frac{-n^2C_0^2(1-\nu^2)}{(E/\rho_f)(h/a)} \quad M_R^* = M_\theta^* - M_R - \frac{\lambda}{n}M_x$$

$\hat{\phi}$ and $\hat{\phi}_R$ at $R = 1$ may be obtained only when the solution to the differential equation is available, subject to the boundary conditions. As the boundary condition at $R = 1$ is reduced to a relationship between $\phi_R$ and $\phi$ (B18), it is only necessary to satisfy the condition at $R = \bar{R}$.

The general solution to the differential Eq. (A2-7) may be written as $\phi = a_1\phi_1 + a_2\phi_2$, where $\phi_1$ and $\phi_2$ are any two independent solutions. $a_1$ and $a_2$ are in general complex. $\phi_1$ and $\phi_2$ may be determined by prescribing the following boundary conditions at $R = \bar{R}$:

$$\phi_1 = 1 \quad \phi_1' = 0 \quad \phi_2 = 0 \quad \phi_2' = 1 \quad \text{(B20)}$$

However, it is clear that the second solution is not admissible as $\phi_2' \neq 0$ at $R = \bar{R}$. Thus the total solution is simply $\phi = a\phi_1$ with

$$\phi(R) = F(R) + iG(R)$$

$$F + iG = (a_{1R} + ia_{1I})(F_1 + iG_1)$$

$$\therefore \begin{Bmatrix}F \\ G\end{Bmatrix}_1 = \begin{bmatrix}a_{1R} & -a_{1I} \\ a_{1O} & a_{1R}\end{bmatrix}_1 \begin{Bmatrix}F_1 \\ G_1\end{Bmatrix} \quad \text{(B21)}$$

$$\begin{Bmatrix}F'(1) \\ G'(1)\end{Bmatrix} = \begin{pmatrix}D_R & -D_I \\ D_I & D_R\end{pmatrix}\begin{Bmatrix}F(1) \\ G(1)\end{Bmatrix} \quad \text{(B22)}$$

Substituting (B21) for $F$ and $G$ reduces (B22) to a system of algebraic equations in $a_{1R}$ and $a_{1I}$. The determinant of the latter provides the condition necessary for the boundary conditions at $R = 1$ to be satisfied. Such a determinant may be shown to be

$$\begin{vmatrix}(F_1' - D_RF_1 + D_IG_1)_1 & (-G_1' + D_IF_1 + D_RG_1) \\ (G_1' - D_IF_1 - D_RG_1)_1 & (F_1' - D_RF_1 - D_IG_1)\end{vmatrix} = 0 \quad \text{(B23)}$$

which is of the form

$$\begin{vmatrix}e_1 & e_2 \\ -e_2 & e_1\end{vmatrix} = 0 \quad \text{(B24)}$$

Clearly this requires that $e_1$ and $e_2$ be zero simultaneously. The numerical procedure is to 1) assume a trial frequency for a given set of parameters, 2) integrate the differential equation with the prescribed boundary conditions at $R = \bar{R}$, and 3) test if $e_1$ and $e_2$ are zero. If not, steps 1–3 are repeated with a new assumed frequency.

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